Linear Algebra

Lecture 1: Vector Spaces

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Vector Spaces

A set V with operations of addition $+: V \times V \to V$ and scalar multiplication $\cdot : \Bbbk \times V \to V$ is a vector space over $\Bbbk$, if for all $v, v_1, v_2, v_3 \in V$ and $\lambda, \mu \in \Bbbk$

- $(V, +)$ is Abelian group and
- $(\lambda\mu)v = \lambda(\mu v)$
- $(\lambda + \mu)v = \lambda v + \mu v$
- $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$
- $1 \cdot v = v.$

Exercises/Examples

- $0 \cdot v = 0$ and $(-1)v = -v$ for any $v \in V$
- $V = 0$, $V = \Bbbk$ are vector spaces
- $V = \Bbbk^n = \{(x_1, x_2, \dots, x_n) \mid x_j \in \Bbbk\}$ is a vector space, where $\lambda(x_1, x_2, \dots, x_n) := (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$ and $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$
- $\mathrm{Mat}_n(\Bbbk)$ is a vector space.

Linear Independence

- a linear combination of $\{v_j\}_{j\in J}$: $\sum_{j\in J}\lambda_j v_j$ (called trivial if all $\lambda_j = 0$)

- a system $\{v_j\}_{j\in J}$ is called *linearly dependent* if there exists a non-trivial linear combination $\sum_{j\in J}\lambda_j v_j = \mathbf{0}$

- Otherwise, it is *linearly independent*.

Basis and Dimension

- A basis of V is maximal linearly independent system
- V is finite dimensional if there exists a finite basis
- If V is finite dimensional then all bases consist of the same number of elements
- This number $\dim V$ is called the dimension of V

Basis and Dimension

- A linear span of a subset $S \subset V$ is a set $\langle S \rangle$ of all finite linear combinations of elements from S
- If $\{e_1, \dots, e_n\}$ is a basis of V, then $V = \langle e_1, \dots, e_n \rangle$
- If $V = \langle e_1, \dots, e_n \rangle$ and $\dim V = n$, then any vector v has a unique representation $v = v_1 e_1 + \dots + v_n e_n$

Coordinates of vectors

- If $V = \langle e_1, \dots, e_n \rangle$, $\dim V = n$, and $v = v_1 e_1 + \dots + v_n e_n$, then numbers $v_1, \dots, v_n$ are called coordinates of a vector v in the basis $\{e_1, \dots, e_n\}$
- Usually we write a vector as a column:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Basis and Dimension: Examples and Exercises

- $\dim \mathbb{R}^n = n$; $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ are its standard basis vectors
- $\dim \mathrm{Mat}_n(\Bbbk) = n^2$ with basis matrices E_{ij} (matrices with 1 at the position (i, j) and zeros anywhere else)
- Vectors $(1, 1)$ and $(1, -1)$ also form a basis of $\mathbb{R}^2$

Linear Maps

- $F\colon V \to W$ is a linear map of vector spaces if $F(a_1 v_1 + a_2 v_2) = a_1 F(v_1) + a_2 F(v_2)$ for any vectors v_1, v_2 and numbers a_1, a_2.
- An isomorphism is a bijective linear map.

Isomorphisms: Lemma

Any n-dimensional space V over $\Bbbk$ is isomorphic to $\Bbbk^n$

Proof: Suppose $V = \langle v_1, \dots, v_n \rangle$ and $\{e_1, \dots, e_n\}$ is the standart basis in $\Bbbk^n$. Then $F(v_j) = e_j$, $j = 1, \dots, n$, defines an isomorphism $F\colon V \to \Bbbk^n$.

Linear Maps and Coordinates

- Suppose $F\colon V \to W$ is a linear map and V has a basis $\{e_1, \dots, e_n\}$
- Then its image $\mathrm{Im}\ F$ is a subspace in W, generated by $F(e_1), \dots, F(e_n)$
- If $v = (v_1, \dots, v_n)^t \in V$, then

$$F(v) = \sum_{k=1}^{n} v_k F(e_k).$$

Linear Maps and Coordinates

- If $v = (v_1, \ldots, v_n)^t \in V$, then

$$F(v) = \begin{pmatrix} \begin{matrix} F(e_1) & \ldots & F(e_n) \end{matrix} \\ \begin{matrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{matrix} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

- It is a matrix form of a map F
- $\dim \operatorname{Im} F = \operatorname{rk} F$

Theorem

Suppose $F\colon V \to W$ is a linear map and $\ker F = \{v \in V \mid F(v) = 0\}$ is its kernel. Then $\dim \operatorname{Im} F + \dim \ker F = \dim V$

Proof: Suppose $\{e_1, \dots, e_k\}$ is a basis of $\ker F$ and $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$ is a basis of V.
Then $\operatorname{Im} F = \langle F(e_{k+1}), \dots, F(e_n)\rangle$ and it remains to prove that these vectors are linearly independent.

Suppose

$$\lambda_1 F(e_{k+1}) + ... + \lambda_{n-k} F(e_n) = 0.$$

Then

$$F(\lambda_1 e_{k+1} + ... + \lambda_{n-k} e_n) = 0,$$

that is, $\lambda_1 e_{k+1} + ... + \lambda_{n-k} e_n \in \ker F$. It is possible iff

$$\lambda_1 = ... = \lambda_{n-k} = 0.$$

Problem 1

(a) Prove that vectors $e_1 = (1, 1)$ and $e_2 = (1, -1)$ form a basis in $\mathbb{R}^2$
(b) Suppose $v_1 = (2, 1)^t$ in the basis $\{e_1, e_2\}$. Find its coordinates in the standard basis.

Solution: (a) $\det(e_1, e_2) = -2 \neq 0$
(b)
$v_1 = 2e_1 + e_2 = 2(1, 1)^t + (1, -1)^t = (3, 1)^t.$

Problem 2

Prove that $\dim \Bbbk[x]_n = n + 1$ and $\Bbbk[x]_n = \langle 1, x, x^2, \dots, x^n \rangle$.

Solution:

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ implies that $\Bbbk[x]_n = \langle 1, x, x^2, \dots, x^n \rangle$;

A system $\{1, x, x^2, \dots, x^n\}$ is linearly independent since $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \equiv 0$ iff all $a_j = 0$.

Problem 3

Prove that $\dim \Bbbk[x] = \infty$ and $\Bbbk[x] = \langle 1, x, x^2, \ldots \rangle$.

Solution: Any

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

implies that $\Bbbk[x] = \langle 1, x, x^2, \ldots \rangle$

We can not restrict the number of basic monomials!

Problem 4

(a) Suppose $\Bbbk$ is a field and $\mathbb{F} \subset \Bbbk$ is its subfield. Then $\Bbbk$ is a vector space over $\mathbb{F}$.
(b) In particular, $\mathbb{C}$ is a 2-dimensional vector space over $\mathbb{R}$.

Solution: (a) Obviously
(b) $\mathbb{C} = \langle 1, i \rangle$, since any $z = a + bi$. And also $a + bi = 0$ iff $a = b = 0$.

Problem 5

Suppose that $\Bbbk$ is a finite field with $\operatorname{char} \Bbbk = p$. Prove that $|\Bbbk| = p^n$ for some number n.

Solution: Obviously, $\mathbb{Z}_p$ is a subfield of $\Bbbk$. Then $\Bbbk$ is a vector space over $\mathbb{Z}_p$. Let $\dim_{\mathbb{Z}_p} \Bbbk = n$. Then $|\Bbbk| = p^n$.

Subspaces

Suppose U, V are subspaces of W. Here are some facts and definitions:

- $U \cap V$ is also a vector space
- $U + V = \{u + v \mid u \in U, v \in V\}$
- A basis of W agrees with U, if U is a span of some basis vectors
- That is, U is a coordinate subspace of W with respect to this basis

Theorem on 2 subspaces

Prove that there exists a basis of W that agrees with subspaces of $U, V \subset W$.

Proof:

Suppose, that $\langle e_1, \dots, e_k \rangle$ is a basis of $U \cap V$, $\langle e_1, \dots, e_k, e_{k+1}, \dots, e_p \rangle$ is a basis of U, and $\langle e_1, \dots, e_k, e_{p+1}, \dots, e_{p+m-k} \rangle$ is a basis of V. Here $\dim(U \cap V) = k$, $\dim U = p$, $\dim V = m$.

Theorem on 2 subspaces: proof

It remains to prove that $\{e_1, \dots, e_{p+m-k}\}$ is a linearly independent system. Then we can complete it to a basis of W. Assume that $\sum_{j=1}^{p+m-k} \lambda_j e_j = 0$. Consider the vector

$$x = \sum_{j=1}^{p} \lambda_j e_j = - \sum_{j=p+1}^{p+m-k} \lambda_j e_j \in U \cap V.$$

It implies that $x = 0$ and all $\lambda_j = 0$.

Problem 6

Suppose U, V are subspaces of W. Prove that

$$\dim(U+V) = \dim U + \dim V - \dim\,(U \cap V)$$

Solution: In the notation of the theorem, the vectors $e_1, \ldots, e_{p+m-k}$ form a basis of $U+V$.
Then $\dim(U+V) = p+m-k$.

Linear Independence of Subspaces

The subspaces $U_1, \dots, U_k$ of V are linearly independent if $u_1 + \dots + u_k = 0$, $u_j \in U_j$, implies that $u_1 = \dots = u_k = 0$.

The following properties are equivalent:

- $U_1, \dots, U_k$ are linearly independent;
- the union of bases of $U_1, \dots, U_k$ is linearly independent;
- $$\dim(U_1 + \dots + U_k) = \dim U_1 + \dots + \dim U_k$$

Direct Sum

A space V is decomposed into the direct sum of its subspaces $U_1, \dots, U_k$ if

- $U_1, \dots, U_k$ are linearly independent;
- $V = U_1 + \dots + U_k$.

We denote it by $V = U_1 \oplus \dots \oplus U_k$.

Decomposition $v = u_1 + \dots + u_k$ is uniquely determined, and u_j is a projection (not orthogonal!) of v on U_j.

Problem 7

- Prove that $\mathbb{R}^2 = U_1 \oplus U_2 := \langle(1,0)\rangle \oplus \langle(1,1)\rangle$;
- Find the projections $(2,2)$ on U_1, U_2.

Solution: The 1st part is easy: the vectors $(1,0)$ and $(1,1)$ form a basis of $\mathbb{R}^2$.
The 2nd part: $(2,2) = 0 \cdot (1,0) + 2(1,1)$.
The projections are $(0,0)$ and $(2,2)$.