Linear Algebra

Lecture 4: Orthogonal Basis for Quadratic Forms

Nikolay V. Bogachev

Moscow Institute of Physics and Technology,
Department of Discrete Mathematics,
Laboratory of Advanced Combinatorics and Network Applicationss

Bilinear Forms

A map $\alpha\colon V \times V \to \mathbb{k}$ is called a bilinear form, if it is linear in both arguments.

Let $\{e_1, \dots, e_n\}$ be a basis of V, and $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be two vectors. Then

$$\alpha(x, y) = \sum_{i,j=1}^{n} a_{ij} x_i y_j = X^T A Y.$$

Symmetric and Skew-Symmetric Forms

α is called symmetric if $\alpha(x, y) = \alpha(y, x)$, and skew-symmetric if $\alpha(x, y) = -\alpha(y, x)$.

It is equivalent to $A^T = A$ and $A^T = -A$, respectively.

A quadratic form associated to symmetric α is $q(x) = \alpha(x, x)$.

Kernel and Non-degenerate Forms

The kernel of α:

$\mathrm{Ker}(\alpha) = \{v \in V \mid \alpha(u, v) = 0 \ \forall u \in V\}$.

α is called non-degenerate if $\mathrm{Ker}(\alpha) = 0$.

Clearly,

$\mathrm{Ker}(\alpha) = \{v \mid \alpha(v, e_j) = 0, \ j = 1, \dots, n\}$.

$\dim \mathrm{Ker}(\alpha) = n - \mathrm{rk}\ A$.

Orthogonal Complement

The orthogonal complement of $U \subset V$ is $U^\perp = \{v \in V \mid \alpha(u,v) = 0 \ \forall u \in U\}$.

Clearly, $V^\perp = \mathrm{Ker}(\alpha)$.

If α is non-degenerate, then

$$\dim U^\perp = \dim V - \dim U \text{ and } (U^\perp)^\perp = U.$$

Non-degenerate subspaces

A subspace $U \subset V$ is non-degenerate with respect to α if $\alpha\,|_U$ is non-degenerate.

$V = U \oplus U^{\perp}$ iff U is non-degenerate.

Orthogonal basis

A basis $\{e_1, \dots, e_n\}$ is orthogonal with respect to α if $\alpha(e_i, e_j) = 0$ for all $i \neq j$.

For any symmetric α there exists an orthogonal basis.

Proof: Induction by $n = \dim V$. If $\alpha \neq 0$, then $q(x) \not\equiv 0$. Then $\exists e_1$: $q(e_1) \neq 0$. Then $V = \langle e_1 \rangle \oplus \langle e_1 \rangle^\perp$, where $\dim \langle e_1 \rangle^\perp = n - 1$.