Linear Algebra

Lecture 7: Linear Operators I

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Linear Operators: Preliminaries

A linear operator in a vector space V is a linear map $\mathcal{A} : V \to V$.

The matrix of an operator $\mathcal{A}$ in a basis $\{e_1, \dots, e_n\}$ is a matrix $A = (a_{ij})$, where $\mathcal{A}(e_j) = \mathcal{A}e_j = \sum_i^n a_{ij} e_i$ (the columns of A).

That is, $(\mathcal{A}e_1, \dots, \mathcal{A}e_n) = (e_1, \dots, e_n)A$.

If $y = \mathcal{A}x$, then $Y = AX$ in the matrix form.

Transition of Coordinates

Let $(e_1', \dots, e_n') = (e_1, \dots, e_n)C$. Then we have

$$(\mathcal{A}e_1', \dots, \mathcal{A}e_n') = (\mathcal{A}e_1, \dots, \mathcal{A}e_n)C =$$

$= (e_1, \dots, e_n)AC = (e_1', \dots, e_n')C^{-1}AC$. Thus,

$$A' = C^{-1}AC\,.$$

Main question: how can we change a basis in such a way that the matrix has a simple form?

Invariant subspaces and eigenvectors are coming!

Invariant Subspaces

A subspace $U \subset V$ is invariant for $\mathcal{A} : V \to \mathrm{V}$ if $\mathcal{A}U \subset U$, i.e. $\mathcal{A}u \in U$ for any $u \in U$.

The restriction $\mathcal{A}|_U$ is an operator in U.

In the basis of V that agrees with U the matrix of $\mathcal{A}$ has the following form:

$\begin{pmatrix} A_0 & B \\ 0 & C \end{pmatrix}$, where $A_0 = \mathrm{Mat}(\mathcal{A}|_U)$

Direct Sum of Invariant Subspaces

If $V = V_1 \oplus \cdots \oplus V_k$, where all V_j are invariant, then $A = \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_k \end{pmatrix}$, where $A_j = \mathrm{Mat}(\mathcal{A}|_{V_j})$.

Simple example: $A = \mathrm{diag}\,(a_1, a_2)$. Here $\mathbb{R}^2 = \langle e_1 \rangle \oplus \langle e_2 \rangle$.

Eigenvectors and Eigenvalues

A non-zero vector $v \in V$ is an eigenvector of $\mathcal{A}$ if $\mathcal{A}v = \lambda v$ for some $\lambda \in \mathbb{F}$ (the field).

The corresponding number $\lambda \in \mathbb{F}$ is called an eigenvalue of $\mathcal{A}$ corresponding to v.

In the basis of eigenvectors $v_1, \ldots, v_n$:

$$A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

Eigenvectors and Eigenvalues

If $\mathcal{A}v = \lambda v$ for some $\lambda \in \mathbb{F}$, then $\langle v \rangle$ is invariant subspace.

Geometrically, eigenvectors are exactly the directions, where the operator acts by stretching of a space by the corresponding eigenvalues.

Characteristic Polynomial

$\mathcal{A}v = \lambda v$ for some $\lambda \in \mathbb{F}$ iff the operator $\mathcal{A} - \lambda I$ is degenerate (singular), that is,

$$\det(A - \lambda E) = 0.$$

The characteristic polynomial of $\mathcal{A}$ is

$$f_{\mathcal{A}}(\lambda) = (-1)^n \det(A - \lambda E).$$

Eigenvalues are exactly the roots of the characteristic polynomial!