Linear Algebra

Lecture 6: Convex Polyhedra I

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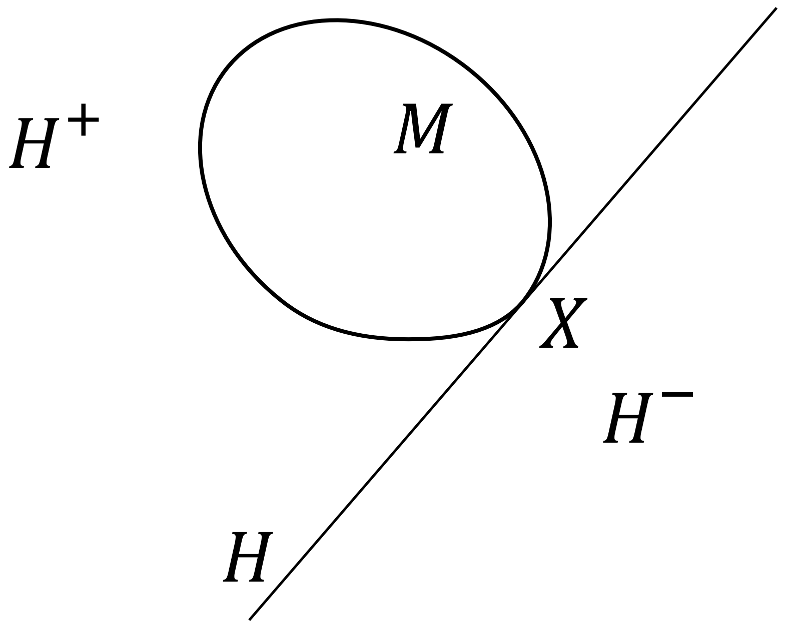
The Separation Theorem

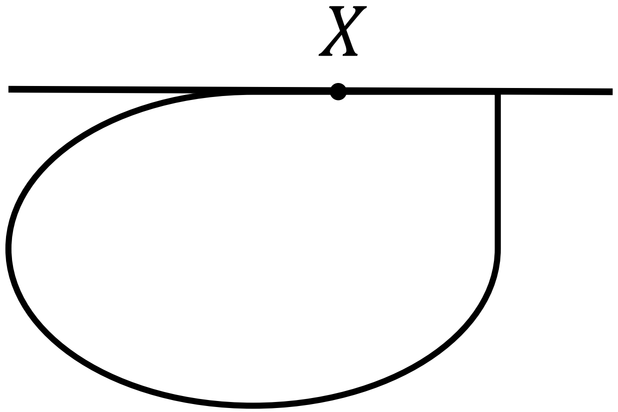
For every point $X \in \partial M$ for a closed convex body $M \subset \mathbb{E}^n$ there exists a supporting hyperplane $H \ni X$.

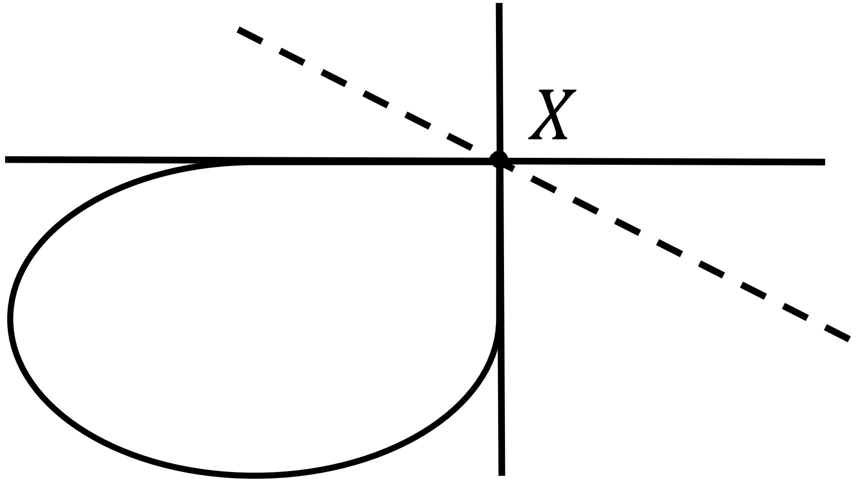


The Separation Theorem

We proved that any plane P trough $X \in \partial M$, s.t. $P \cap \text{int}(M) = \emptyset$, is contained in a supporting hyperplane.

$X \in \partial M$ can belong to either a unique or infinitely many supporting hyperplanes.





Intersection of Half-Spaces

Every closed convex set is an intersection of (perhaps infinitely many) half-spaces.

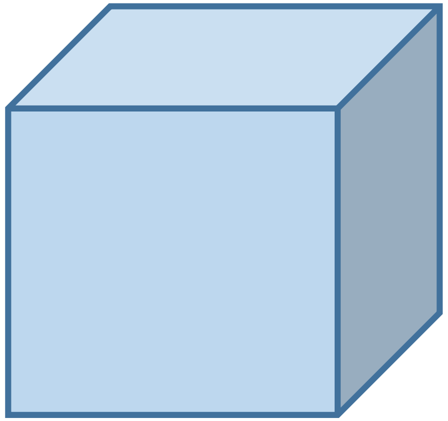
Proof: $H_f = H_f^+ \cap H_f^-$, it implies that any plane is an intersection of half-spaces.

Thus, it remains to prove the theorem for a convex body.

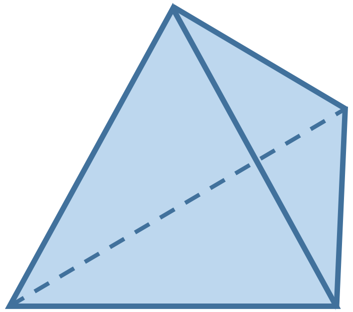
Every convex body is the intersection of half-spaces of its supporting hyperplanes.

Polyhedron

A convex polyhedron is an intersection of finitely many half-spaces (sometimes, non-empty interior is required).



Parallelepiped



Simplex

Extreme Points

A point $A \in M$ for a convex M is extreme if it is not an interior point of any interval in M.

Theorem. A bounded closed convex set M is the convex hull of the set $E(M)$ of its extreme points.

Proof: Let $\widetilde{M} = \operatorname{conv} E(M)$. Clearly, $\widetilde{M} \subset M$. We will prove by induction on $n = \dim \mathbb{E}^n$ that $M \subset \widetilde{M}$. Assume that $n > 0$, $A \in M$, and M is a convex body.

Extreme Points

Proof: Assume that $n > 0$, $A \in M$, and M is a convex body. We'll prove that $A \in \widetilde{M}$.

Case 1: $A \in \partial M$. Taking a supporting hyperplane $H \ni A$, we obtain that a bounded closed convex set $H \cap M = \operatorname{conv} E(H \cap M)$ and $A \in \widetilde{M}$.

Case 2: $A \in \operatorname{int} M$. Then $A \in (X, Y)$, where $X, Y \in \partial M$, and therefore, $X, Y \in \widetilde{M}$.

Thus, $A \in \widetilde{M}$.

Minkowski-Weyl Theorem

M is a convex polyhedron iff M is a convex hull of finitely many points.

Proof: Let $M = \bigcap_{j=1}^{m} H_{f_j}^{+}$ be a convex polyhedron. Let us prove that $\forall X \in E(M)$ is the only point in the intersection of some of $H_{f_1}^{+}, \dots, H_{f_m}^{+}$.

This will imply that

$$\#\left(E(M)\right) < +\infty, \text{ and } M = \operatorname{conv}\left(E(M)\right).$$

Minkowski-Weyl Theorem

Proof: Let $A \in E(M)$. Define

$$J = \{\, j \mid f_j(A) = 0 \,\} \subset \{1, \dots, m\},$$

$$P = \{\, X \in \mathbb{E}^n \mid f_j(X) = 0,\ j \in J \,\}.$$

Since $f_k(A) > 0$ for $k \notin J$, we see that $A \in \operatorname{int}(M \cap P)$ in the space P.

But $A \in E(M)$, hence $A \in E(M \cap P)$. Thus, $\dim P = 0$, that is, $P = \{A\}$.

Minkowski-Weyl Theorem

Proof: Let $M = \text{conv}\{A_1, \dots, A_k\}$. We assume that $\text{aff}(M) = \mathbb{E}^n$. Consider

$$M^* = \{f \mid f(A_j) \geq 0 \text{ for } 1 \leq j \leq k, \sum_{j=1}^{k} f(A_j) = 1\}$$

Any f is uniquely determined by $f(A_j)$ for $j = 1, \dots, k$. Since $|f(A_j)| \leq 1$, then M^* is bounded and $M^* = \text{conv}\{f_1, \dots, f_m\}$. Thus,

$$M = \{X \in \mathbb{E}^n \mid f(X) \geq 0 \ \forall f \in M^*\} =$$

$$= \{X \in \mathbb{E}^n \mid f_k(X) \geq 0 \ \forall k = 1, \dots, m\}.$$