ARITHMETICITY OF IDEAL HYPERBOLIC RIGHT-ANGLED POLYHEDRA AND HYPERBOLIC LINK COMPLEMENTS

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Abstract. In this paper we provide a generalized construction of nonarithmetic hyperbolic orbiifolds in the spirit of Gromov and Piatetski-Shapiro. Nonarithmeticity of such orbifolds is based on recently obtained results connecting a behaviour of the so-called fc-subspaces (totally geodesic subspaces fixed by finite order elements of the commensurator) with arithmetic properties of hyperbolic orbifolds and manifolds.

As an application, we verify arithmeticity of some particular class of ideal hyperbolic right-angled 3-polyhedra and hyperbolic link complements.

1. Introduction

In recent years, there has been growing interest in the study of geometric and arithmetic properties of hyperbolic right-angled polyhedra, in large part due to their connection to hyperbolic link complements. In this paper we provide some tools and use them to verify arithmeticity of some particular class of ideal hyperbolic right-angled polyhedra. We also check on arithmeticite one infinite family of link complements.

Our first main result is based on the ideas of Gromov and Piatetski-Shapiro [GPS87] for constructing nonarithmetic hyperbolic lattices (including ones generated by reflections, see also Vinberg [Vin14]) and uses results from a recent paper [BBKS] by Belolipetsky et al.

Let $\Gamma_1, \Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$ be two lattices both containing a reflection r in a hyperplane $H \subset \mathbb{H}^n$ such that:

(1) for any $\gamma \in \Gamma_i$, $i = 1, 2$, either $\gamma(H) = H$ or $\gamma(H) \cap H = \emptyset$,
(2) $N_{\Gamma_1}(r) = N_{\Gamma_2}(r) = \langle r \rangle \times \Gamma_0$, where Γ_0 leaves invariant the two half–spaces bounded by H. Here $N_{\Gamma_i}(r)$ denotes the normalizer of r in Γ_i.

This means that H projects to the same embedded totally geodesic subspace in both finite volume hyperbolic orbifolds $\mathcal{O}_1 = \mathbb{H}^n/\Gamma_1$ and $\mathcal{O}_2 = \mathbb{H}^n/\Gamma_2$.

Consider the set $\Omega_i = \{\gamma(H) \mid \gamma \in \Gamma_i\}$ for a fixed $i = 1$ or 2. The set Ω_i decomposes the space $\mathbb{H}^n$ into a collection of closed domains transitively permuted by the group Γ_i, and each of these pieces is a fundamental domain for the action of a reflection group $N_i = \langle \gamma r \gamma^{-1} \mid \gamma \in \Gamma_i \rangle$, which is the normal closure of r in Γ_i. Let D_i be one of these pieces, and let $\Delta_i = \{\gamma \in \Gamma_i \mid \gamma(D_i) = D_i\}$. Then the group Γ_i can be decomposed into a semidirect product $\Gamma_i = N_i \rtimes \Delta_i$. Moreover, by choosing D_1 and D_2 to belong to opposite half-spaces with respect to the hyperplane H, we can ensure that $\Delta_1 \cap \Delta_2 = \Gamma_0$ is the common stabilizer of the hyperplane H in both lattices Γ_1 and Γ_2.

Then the following result shows that we can build a new "hybrid" non-arithmetic lattice out of Γ_1 and Γ_2 described above.

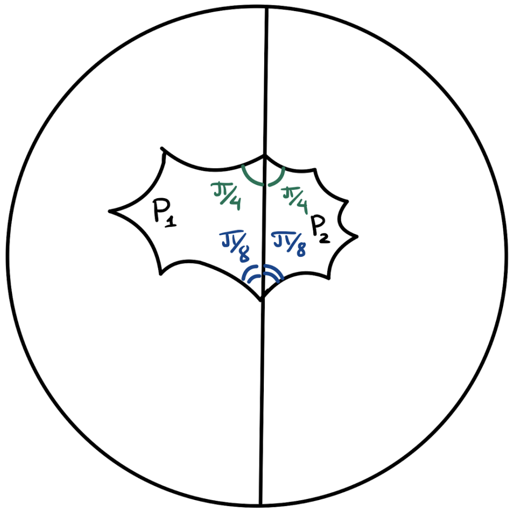

FIGURE 1. This is an illustration for Theorem 1.2. A Coxeter polygon P glued from two other Coxeter polygons P_1 and P_2 having a common side F. This side F has angles $\pi/4$ and $\pi/8$ with adjacent sides in both polygons.

Theorem 1.1 (Generalised Gromov—Piatetski-Shapiro nonarithmetic orbifolds)**.** *The group* $\Gamma = \langle \Delta_1, \Delta_2 \rangle = \Delta_1 *_{\Gamma_0} \Delta_2 < \mathrm{Isom}(\mathbb{H}^n)$ *is a lattice. If* Γ_1 *and* Γ_2 *are incommensurable, then the lattice* Γ *is nonarithmetic.*

In the terminology of geometric topology, this roughly means that we glue a finite volume hyperbolic orbifold $\mathcal{O} = \mathbb{H}^n/\Gamma$ from two finite volume hyperbolic orbifolds $\mathcal{O}_1'$ and $\mathcal{O}_2'$ with common totally geodesic boundary which is fixed pointwise by the reflection $r \in \Gamma_1 \cap \Gamma_2$. Let us illustrate this construction in the setting of reflection orbifolds and hyperbolic Coxeter polyhedra.

Theorem 1.2. *Let* P_1 *and* P_2 *be two finite volume Coxeter polyhedra in* $\mathbb{H}^n$ *such that associated reflection groups* Γ_{P_1} *and* Γ_{P_2} *(generated by reflections in all of the walls of* P_1 *and* P_2*, respectively) are not commensurable. Suppose that* P_1 *and* P_2 *have a common facet* F *which meets all of its adjacent facets both in* P_1 *and* P_2 *at even angles, i.e. angles of the form* $\pi/2m$*, such that respective adjacent angles with respect to the common facet* F *are equal for* P_1 *and* P_2 *(see Figure 1).*

Then the polyhedron P *obtained by glueing* P_1 *with* P_2 *along* F *is also a Coxeter polyhedron and is associated to a non-arithmetic hyperbolic reflection group* Γ_P*.*

In 1991, Reid proved [Reid] that the figure-eight knot complement is the only knot complement being an arithmetic hyperbolic 3-manifold. On the other hand, the classification of arithmetic link complements is still far from being completed.

A lot of hyperbolic links, in particular, the *fully augmented links* are associated to *ideal right-angled polyhedra* in $\mathbb{H}^3$. Fundamental groups of such link complements are not always commensurable with associated right-angled reflection groups, but in many cases they are. Thus, it becomes important to classify arithmetic ideal hyperbolic right-angled 3-dimensional polyhedra.

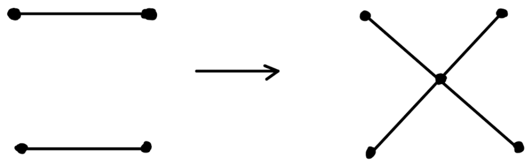

FIGURE 2. The edge twist.

Ideal right-angled antiprisms A_n constitute an especially interesting subclass of ideal hyperbolic right-angled polyhedra. In particular, due to the fact that any ideal hyperbolic right-angled polyhedron can be from one of the antiprisms by some number of the so-called *edge twist* operations. This edge twist operation is illustrated in Figure 2. This is a combinatorial operation: we take two disjoint edges belonging to one face, remove them, then add a new vertex and join this vertex with other vertices of the removed edges. The resulting combinatorial polyhedron can always be realized in $\mathbb{H}^3$ as an ideal right-angled polyhedron by Andreev's theorem. Generally, it is quite hard to say what is the geometry of a constructed polyhedron.

However, in some certain situations we are able even to say something about arithmeticity of twisted antiprisms.

Theorem 1.3. *Let $A_{n,k}$ be an ideal hyperbolic right-angled polyhedron obtained from the antiprism A_n by twisting two edges in one of its n-gonal faces such that the there are $k-2$ edges between twisted edges, where $3 \leqslant k \leqslant n/2+1$. Then $A_{n,k}$ is glued from A_k and A_{n-k+2} along their common ideal triangular face, and it is associated to an arithmetic right-angled reflection group if and only if $n = 6$, $k = 4$, or $n = 4$, $k = 3$.*

Recently, Meyer–Millichap–Trapp [MMT20] and Kellerhals [Kel22] showed that the ideal right-angled antiprisms A_n provide a sequence of pairwise incommensurable reflection groups, and proved that the corresponding reflection groups are arithmetic if and only if $n = 3$ or 4. This implies that the complement $M_n = \mathbb{S}^3 \backslash \mathcal{D}_{2n}$ of the classical chain link $\mathcal{D}_{2n}$ is an arithmetic hyperbolic 3-manifold only for these values of n.

Theorem 1.4. *Let $\mathcal{C}_{4n+1}$ be a link obtained from the classical chain link $\mathcal{D}_{4n}$ by adding one more diagonal ring in such a way that it corresponds to a reflective symmetry of $\mathcal{D}_{4n}$; see Figure 3 for the link $\mathcal{C}_{13}$. Then the fundamental group Γ_{4n+1} of its complement $N_{4n+1} = \mathbb{S}^3 \setminus \mathcal{C}_{4n+1}$ is an arithmetic hyperbolic lattice if and only if $n = 2$ or $n = 3$. Moreover, Γ_{4n+1} and Γ_{4m+1} are commensurable if and only if $m = n$.*

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2. PRELIMINARIES ON HYPERBOLIC POLYHEDRA, KNOTS AND LINKS

2.1. **Convex hyperbolic polyhedra.** Let $\mathbb{R}^{d,1}$ be the real vector space $\mathbb{R}^{d+1}$ equipped with the standard Lorentzian scalar product of signature $(d, 1)$, namely,

$$(x, y) = -x_0y_0 + x_1y_1 + \cdots + x_dy_d.$$

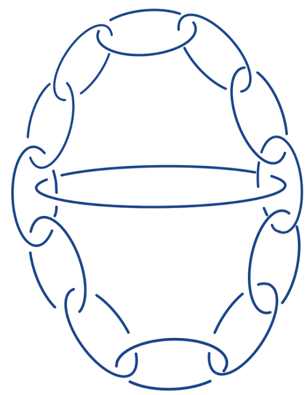

FIGURE 3. The arithmetic link $\mathcal{C}_{13}$.

The hyperboloid $\mathcal{H} = \{x \in \mathbb{R}^{d,1} \,|\, (x,x) = -1\}$ has two connected components

$$\mathcal{H}^+ = \{x \in \mathcal{H} \,|\, x_0 > 0\} \text{ and } \mathcal{H}^- = \{x \in \mathcal{H} \,|\, x_0 < 0\}.$$

The d-dimensional hyperbolic space $\mathbb{H}^d$ is the Riemannian manifold $\mathcal{H}^+$ with the metric ρ induced by restricting (x,y) to tangent bundle $T\mathcal{H}^+$. This hyperbolic metric ρ satisfies $\cosh \rho(x,y) = -(x,y)$. The hyperbolic d-space $\mathbb{H}^d$ is known to be the unique simply connected complete Riemannian d-manifold with constant sectional curvature -1. *Hyperplanes* of $\mathbb{H}^d$ are intersections of linear hyperplanes of $\mathbb{R}^{d,1}$ with $\mathcal{H}^+$, and are totally geodesic submanifolds of codimension 1 in $\mathbb{H}^d$.

Let $\mathrm{O}_{d,1} = \mathbf{O}(f, \mathbb{R})$ be the orthogonal group of the form f, and $\mathrm{O}'_{d,1} < \mathrm{O}_{d,1}$ be the subgroup (of index 2) preserving $\mathcal{H}^+$. The group $\mathrm{O}'_{d,1}$ preserves the metric ρ on $\mathbb{H}^d$, and is in fact the full group $\mathrm{Isom}(\mathbb{H}^d)$ of isometries of the latter.

If $\Gamma < \mathrm{O}'_{d,1}$ is a lattice, i.e., if Γ is a discrete subgroup of $\mathrm{O}'_{d,1}$ with a finite-volume fundamental polyhedron in $\mathbb{H}^d$, then the quotient $M = \mathbb{H}^d/\Gamma$ is a complete finite-volume *hyperbolic orbifold*. If Γ is torsion-free, then M is a complete finite-volume Riemannian manifold, and is called a *hyperbolic manifold*.

Now set $G = \mathrm{O}'_{d,1}$, and suppose $\mathbf{G}$ is an admissible (for G) algebraic k-group, i.e. $\mathbf{G}(\mathbb{R})^o$ is isomorphic to G^o and $\mathbf{G}^\sigma(\mathbb{R})$ is a compact group for any non-identity embedding $\sigma \colon k \hookrightarrow \mathbb{R}$. Then any subgroup $\Gamma < G$ commensurable with the image in G of $\mathbf{G}(\mathcal{O}_k)$ is an *arithmetic lattice* (in G) with *ground field* k.

In the hyperboloid model, points from the ideal hyperbolic boundary correspond to isotropic vectors:

$$\partial \mathbb{H}^d = \{x \in \mathbb{R}^{d,1} \mid (x,x) = 0 \text{ and } x_0 > 0\}/\mathbb{R}_+.$$

A *convex hyperbolic d-polyhedron* is the intersection, with non-empty interior, of a finite family of closed half-spaces in hyperbolic d-space $\mathbb{H}^d$. A *hyperbolic Coxeter n-polyhedron* is a convex hyperbolic d-polyhedron P all of whose dihedral angles are integer sub-multiples of π, i.e. of the form π/m for some integer $m \geqslant 2$. A hyperbolic Coxeter polyhedron is called *right-angled* if all its dihedral angles are

$\pi/2$. A generalized[1] convex polyhedron is said to be *acute-angled* if all its dihedral angles do not exceed $\pi/2$.

It is known that generalized Coxeter polyhedra are the natural fundamental domains of discrete groups genereted by reflections in spaces of constant curvature, see [Vin85].

A convex d-polyhedron has *finite volume* if and only if it is the convex hull of finitely many points of the closure $\overline{\mathbb{H}^d} = \mathbb{H}^d \cup \partial\mathbb{H}^d$. A convex polyhedron is said to be *ideal*, if all its vertices are *ideal*, i.e. belong to the ideal hyperbolic boundary $\partial\mathbb{H}^d$.

Two compact polytopes P and P' in Euclidean space $\mathbb{E}^n$ are *combinatorially equivalent* if there is a bijection between their faces that preserves the inclusion relation. A combinatorial equivalence class is called a *combinatorial polytope*. Note that if a hyperbolic polyhedron $P \subset \mathbb{H}^n$ is of finite volume, then the closure $\overline{P}$ of P in $\overline{\mathbb{H}^n}$ is combinatorially equivalent to a compact polytope of $\mathbb{E}^n$.

The following theorem is a special, right-angled, case of Andreev's theorem (see [And71a, And71b]).

Theorem 2.1. *Let $\mathcal{P}$ be a combinatorial 3-polytope. There exists a finite–volume right-angled hyperbolic 3-polyhedron $P \subset \overline{\mathbb{H}^3}$ that realizes $\mathcal{P}$ if and only if:*

(1) $\mathcal{P}$ is neither a tetrahedron, nor a triangular prism;
(2) every vertex of P belongs to at most four faces;
(3) if f, f', and f'' are faces of $\mathcal{P}$, and $e' = f \cap f'$, $e'' = f \cap f''$ are non-intersecting edges, then f' and f'' do not intersect each other;
(4) there are no faces f_1, f_2, f_3, f_4 such that $e_i := f_i \cap f_{i+1}$ (indices $\bmod 4$) are pairwise non-intersecting edges of $\mathcal{P}$.

In a right-angled polyhedron $P \subset \overline{\mathbb{H}^3}$, a vertex v lies in $\mathbb{H}^3$ (i.e. is finite) if and only if it belongs to exactly three faces of P. If a vertex v is contained in four faces of P, then it is ideal, i.e. $v \in \partial\mathbb{H}^3$.

2.2. **Arithmeticity and finite centraliser subspaces.** Arithmetic properties of finite volume hyperbolic d-orbifolds are connected with a behaviour of the so-called fc-subspaces. Let $\mathrm{Comm}(\Gamma)$ denote the commensurator of Γ in $G = \mathrm{O}'_{d,1}$. For a given froup $F < \mathrm{Comm}(\Gamma)$, let $\mathrm{Fix}(F) = \{x \in \mathbb{H}^d \mid gx = x,\ \forall g \in F\}$ is the fixed point set of F in $\mathbb{H}^d$.

Definition 2.2. *An immersed totally geodesic suborbifold N of a hyperbolic orbifold $M = \mathbb{H}^d/\Gamma$ is called a finite centraliser subspace (an fc–subspace for short) if there exists a finite subgroup $F < \mathrm{Comm}(\Gamma)$ such that $H = \mathrm{Fix}(F)$ is a totally geodesic subspace of $\mathbb{H}^d$ and $N = H/\mathrm{Stab}_\Gamma(H)$.*

There are three types of lattices in G that we shall distinguish. *Type I* arithmetic lattices come from admissible quadratic forms, *type II* arithmetic lattices come from skew–Hermitian forms over quaternion algebras, and *type III* arithmetic lattices comprise one exceptional family in $\mathbf{PO}_{3,1}(\mathbb{R})$ and another in $\mathbf{PO}_{7,1}(\mathbb{R})$ (the so-called "trialitarian lattices").

The central result of paper [BBKS] is the following theorem.

[1]A *generalized convex polyhedron* P is the intersection, with non-empty interior, of possibly infinitely many closed half-spaces in hyperbolic d-space such that every closed ball intersects only finitely many bounding hyperplanes of P

Theorem 2.3. *Let $M = \mathbb{H}^d/\Gamma$ be a finite volume hyperbolic n–orbifold. We have:*

(1) If M is arithmetic, then it contains infinitely many fc–subspaces of positive dimension. Moreover, all totally geodesic suborbifolds of M of dimension $m \geqslant \lfloor \frac{d}{2} \rfloor$ which are not 3–dimensional type III are fc–subspaces.

(2) If M is non–arithmetic, then it has finitely many fc–subspaces, their number being bounded above by $c \cdot \mathrm{Vol}(M)$, with a constant $c = const(d)$ depending only on d.

3. Proof of main results

3.1. **Proof of Theorem 1.1.** Note that the hyperplane H projects to a totally geodesic subspace H/Γ_0 in the orbifold $\mathbb{H}^n/\Gamma$. Let us prove that this is not an fc-subspace. Arguing by contradiction, we assume that the reflection r in the hyperplane H commensurates the lattice Γ. By setting $\Gamma' = r\Gamma r^{-1} \cap \Gamma$ and $\Gamma'' = \langle \Gamma', r\rangle$, we obtain that Γ'' is commensurable with Γ.

Let $\Delta_i' = \Delta_i \cap \Gamma'$, and $\Gamma_0' = \Gamma_0 \cap \Gamma'$. The normaliser in Γ'' of the reflection r is the group $N_{\Gamma''}(r) = \langle r\rangle \times \Gamma_0' = \langle r\rangle \times (\Delta_1' \cap \Delta_2')$. Let us denote by N' the normal closure of r in Γ'' (notice that N' is generated by reflections $\gamma r \gamma^{-1}$ for all $\gamma \in \Gamma''$), and set $\Gamma_i' = N' \rtimes \Delta_i'$. It is clear that Δ_i' is a finite index subgroup of Δ_i, and that Γ_i' is a finite index subgroup of Γ_i. Moreover, both Γ_1' and Γ_2' are sublattices of Γ'' and therefore have finite index in Γ''. This implies that Γ_1 and Γ_2 are commensurable, which is a contradiction.

Since H/Γ_0 is a codimension 1 non-fc subspace of the orbifold $\mathcal{O} = \mathbb{H}^n/\Gamma$, the main theorem from [BBKS] implies that the lattice Γ is not arithmetic.

3.2. **Proof of Theorem 1.2.** Theorem 1.2 is a special case of Theorem 1.1: Δ_i is just the group generated by reflections in all the walls of P_i excepting the supporting hyperplane H of the facet F. Then it remains to see from the contruction, that Γ_0 is generated by reflections in hyperplanes that are orthogonal to H.

Here we explicitely use the fact that F meets its adjacent facets at even angles. Indeed, let H and the supporting hyperplane H_j' of a facet F_j' of P_j form the same angle $\pi/2m$ for $j = 1, 2$. Then the stabiliser of $H \cap H_j'$ in both Γ_{P_1} and Γ_{P_2} is the dihedral group of order $2m$. This implies that there exists a reflection r'' in $\Gamma_{P_1} \cap \Gamma_{P_2}$ with mirror H'' orthogonal to H. This reflection r'' leaves H invariant, and thus $r'' \in \Gamma_0$. This implies that the stabiliser Γ_0 contains all reflections whose mirrors (intersected with H) bound the facet F inside the hyperplane H. On the other hand, it is clear that the fundamental domain Γ_0 in $H \simeq \mathbb{H}^{n-1}$ contains F, and thus they coincide, and Γ_0 is indeed generated by reflections in hyperplanes orthogonal to H.

3.3. **Proof of Theorem 1.3.** The first of the theorem is to prove that $A_{n,k}$ is glued from A_k and A_{n-k+2} along their common ideal triangular face. This can be easily obtained by drawing the combinatorial diagram of $A_{n,k}$, then cutting it along a plane corresponding to a common triangular face and applying Andreev's theorem for right-angled polyhedra.

Then Theorems 1.1 and 1.2 together with results from [MMT20, Kel22] imply that $A_{n,k}$ is arithmetic if and only if A_k and A_{n-k+2} coincide (since A_n and A_m are not commensurable for $m \neq n$) and both are arithmetic, which actually means that $k = 3 = n - k + 2$ or $k = 4 = n - k + 2$. This gives us only $n = 4$, $k = 3$, and $n = 6$, $k = 4$.

3.4. **Proof of Theorem 1.4.** First of all, let us observe that the links $\mathcal{C}_{4n+1}$ are fully augmented links decomposable into two equal copies of the ideal right-angled polyhedron $A_{2n,n+1}$ in $\mathbb{H}^3$. Combinatorially, these polyhedra are just twisted antiprisms A_{2n} where the twisting is applied to the pair of opposite non-adjacent edges of the $2n$-gonal facet. By Theorem 1.3, this polyhedron is glued from two equal antiprisms A_{n+1}.

Due to [CDBW12, Lemma 6.1], $\Gamma_{4n+1} = \pi_1 N_{2n+1}$ is commensurable with a group generated by reflections in the walls of $A_{2n,n+1}$, which is commensurable with the reflection group of A_{n+1}. The latter is known [MMT20, Kel22] to be arithmetic only for $n+1 = 3, 4$. Also, we get that Γ_{4n+1} and Γ_{4m+1} are commensurable if and only if A_{4n+1} and A_{4m+1} are commensurable, and again by [MMT20, Kel22] this is possible if and only if $n = m$.

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